A REMARK ON SOLITON EQUATION OF MEAN CURVATURE FLOW

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Abstract. In this short note, we consider self-similar immersions $F : \mathbb{R}^n \to \mathbb{R}^{n+k}$ of the Graphic Mean Curvature Flow of higher co-dimension. We show that the following is true: Let $F(x) = (x, f(x)), x \in \mathbb{R}^n$ be a graph solution to the soliton equation

$$\overline{H}(x) + F^{\perp}(x) = 0.$$

Assume $\sup_{\mathbb{R}^n} |Df(x)| \leq C_0 < +\infty$. Then there exists a unique smooth function $f_\infty : \mathbb{R}^n \to \mathbb{R}^k$ such that

$$f_\infty(x) = \lim_{\lambda\to\infty} f_\lambda(x)$$

and

$$f_\infty(rx) = rf_\infty(x)$$

for any real number $r \neq 0$, where

$$f_\lambda(x) = \lambda^{-1} f(\lambda x).$$

1. Introduction

Let M^{n+k} be a Riemannian manifold of dimension $n + k$. Assume that Σ^n be a Riemannian manifold of dimension n without boundary. Let $F : \Sigma^n \to M^{n+k}$ be an isometric immersion. Denote ∇ (respectively D) the covariant differentiation on Σ (on M). Let $T\Sigma$ and $N\Sigma$ be the tangent bundle and normal bundle of Σ in M respectively. We define the second fundamental form of the immersion Σ by

$$II : T\Sigma \times T\Sigma \to N\Sigma,$$

with

$$II(X, Y) = D_X Y - \nabla_X Y,$$

for tangential vector fields X, Y on Σ. We define the mean curvature vector field (in short, MCV) by

$$\overline{H} = \mathrm{tr}_\Sigma II.$$

In recent years, many people are interested in studying the evolution of the immersion $F : \Sigma^n \to M^{n+k}$ along its Mean Curvature Flow (in short, just say MCF). The MCF is defined as follows. Given an one-parameter family of sub-manifolds $\Sigma_t = F_t(\Sigma)$ with immersions $F_t : \Sigma \longrightarrow M$. Let $\overline{H}(t)$ be the MCV of Σ_t. Then our MCF is the equation/system

$$\frac{\partial F(x,t)}{\partial t} = \overline{H}(x,t).$$

This flow has many very nice results if the codimension $k = 1$. See the work of G.Huisken [3] for a survey in this regard. Since there is very few result about MCF in higher codimension, we will study it in the target when $M^{n+k} = R^{n+k}$, which is the standard Euclidian space.

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In this short note, we will consider a family of self-similar graphic immersions $F(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^{n+k}$ of the Mean Curvature Flow (MCF):

$$\frac{\partial}{\partial t} F(x,t) = \overline{H}(x,t), \quad \forall x \in \mathbb{R}^n, \quad \forall t \in (-\infty, 0).$$

Write

$$\Sigma_t = F(\mathbb{R}^n, t),$$

and

$$F = (F^A), \quad 1 \leq A \leq n+k.$$

By definition, we call the family Σ_t *self-similar* if

$$\Sigma_t = \sqrt{-t}\Sigma_{-1}, \quad \forall t < 0.$$

In this case, we can reduce the MCF into an elliptic system. In the other word, we have the following parametric elliptic equation for the family Σ_t:

$$\overline{H}(x) + F^{\perp}(x) = 0, \quad \forall x \in \Sigma_{-1} := \Sigma.$$

We will call this system as the *soliton equation* of the MCF. Note that this equation is usually obtained from the monotonicity formula of G.Huisken [2] for blow-up. It is a hard and open problem to classify solutions of this equation.

Fix $\Sigma = \Sigma_t$. Assume that $F(x) = (x, f(x))$. Let

$$Q = (Q^A_\alpha), \ \ n+1 \leq \alpha \leq n+k \ \ 1 \leq A \leq n+k$$

is the orthogonal projection onto $N_p\Sigma$, where $p \in \Sigma$. Then the second fundamental form of Σ can be written as

$$\mathrm{II}^A_{ij} = Q^A_\alpha D^2_{ij} f^\alpha.$$

Hence, we have the expression for the mean curvature vector of Σ in $\mathbb{R}^{n+k}$:

$$\overline{H}^A = g^{ij} Q^A_\alpha D^2_{ij} f^\alpha.$$

Our main result in this paper is the following

Theorem 1.1. *Let $F(x) = (x, f(x)), x \in \mathbb{R}^n$ be a graph solution to the soliton equation*

$$\overline{H}(x) + F^{\perp}(x) = 0.$$

Assume $\sup_{\mathbb{R}^n} |Df(x)| \leq C_0 < +\infty$. Then there exists a unique smooth function $f_\infty : \mathbb{R}^n \to \mathbb{R}^k$ such that

$$f_\infty(x) = \lim_{\lambda \to \infty} f_\lambda(x)$$

and

$$f_\infty(rx) = r f_\infty(x)$$

for any real number $r \neq 0$, where

$$f_\lambda(x) = \lambda^{-1} f(\lambda x).$$

We remark that the proof of this result given below is very simple. But it is based on a nice observation. We just use the divergence theorem with a nice test function. In the next section, we recall the form of divergence theorem for convenient of the readers. In the last section we give a proof of our Theorem.

We point out that we may consider $F_\infty(x) = (x, f_\infty(x))$ obtained above as a tangential minimal cone along the research direction done by L.Simon [5].

2. Preliminary

Given a vector field $X:\Sigma\to TM$. Let X^T and X^N denote the projection of X onto $T\Sigma$ and $N\Sigma$ respectively. We define the divergence of X on Σ as

$$\mathrm{div}_\Sigma X=\sum g^{ij}\langle D_iX,\frac{\partial}{\partial x^j}\rangle$$

where $(g^{ij})=g_{ij}^{-1}$, and (g^{ij}) is the induced metric tensor written in local coordinates (x^i) on Σ.

Note that, for any tangential vector field Y on Σ,

$$D_YX=D_YX^T+D_YX^N.$$

So

$$\begin{aligned}\langle D_YX,Y\rangle&=\langle D_YX^T,Y\rangle+\langle D_YX^N,Y\rangle\\&=\langle\nabla_YX^T,Y\rangle-\langle D_YY,X^N\rangle\\&=\langle\nabla_YX^T,Y\rangle-\langle II(Y,Y),X\rangle.\end{aligned}$$

Hence

$$\mathrm{div}_\Sigma X^T=\mathrm{div}_\Sigma X+\langle X,\overline{H}\rangle,$$

and by the Stokes formula on Σ, we have

$$\int_\Sigma \mathrm{div}X^T=\int_{\partial\Sigma}\langle X,\nu\rangle d\sigma$$

and

$$\int_\Sigma \mathrm{div}_\Sigma X d\nu=-\int_\Sigma\langle\overline{H},X\rangle d\nu+\int_{\partial\Sigma}\langle X,\nu\rangle d\sigma,$$

where ν is the exterior normal vector field to Σ on $\partial\Sigma$.

3. Proof of Main Theorem

In the following, we take $M^{n+k}=\mathbb{R}^{n+k}$ as the standard Euclidean space. We assume that the assumption of our Theorem 1.1 is true in this section.

Define the vector field

$$X=(1+|F|)^{-s}F$$

where $s\in\mathbb{R}$ to be determined.

Note that, $\nabla|F|=\frac{F^\top}{|F|}$ and $\mathrm{div}_\Sigma F=n$. So

$$\begin{aligned}\mathrm{div}_\Sigma X&=\langle\nabla(1+|F|)^{-s},F\rangle+(1+|F|)^{-s}\mathrm{div}_\Sigma F\\&=-\frac{s(1+|F|)^{-s-1}}{|F|}|F^\top|^2+n(1+|F|)^{-s}.\end{aligned}$$

Locally, we may assume that Σ is a graph of the form $(x,f(x))\in B_R(0)\times\mathbb{R}^k$, where $B_R(0)$ is the ball of radius R centered at 0. Let $\Sigma_R=\Sigma\cap(B_R(0)\times\mathbb{R}^k)$. By the divergence theorem we have

$$\int_{\Sigma_R}\mathrm{div}_\Sigma X=\int_{\Sigma_R}\langle\overline{H},X\rangle-\int_{\partial\Sigma_R}\langle X,\nu\rangle$$

By direct computation, we have that

$$\begin{aligned}\int_{\Sigma_R} \mathrm{div}_\Sigma X &= -s\int_{\Sigma_R}\frac{(1+|F|)^{-s-1}}{|F|}|F^\top|^2 + n\int_{\Sigma_R}(1+|F|)^{-s}\\ &= -\int_{\Sigma_R}(1+|F|)^{-s}|F^\perp|^2 - \int_{\partial\Sigma_R}(1+|F|)^{-s}\langle F,\nu\rangle\\ &= -\int_{\Sigma_R}(1+|F|)^{-s}|\overline{H}|^2 - \int_{\partial\Sigma_R}(1+|F|)^{-s}\langle F,\nu\rangle.\end{aligned}$$

Hence, we have

$$\int_{\Sigma_R}(1+|F|)^{-s}|\overline{H}|^2 = s\int_{\Sigma_R}\frac{(1+|F|)^{-s-1}}{|F|}|F^\top|^2 - n\int_{\Sigma_R}(1+|F|)^{-s} - \int_{\partial\Sigma_R}(1+|F|)^{-s}\langle F,\nu\rangle.$$

Since $|F^\top| \le |F| \le 1+|F|$, we have

$$\int_{\Sigma_R}\frac{(1+|F|)^{-s-1}}{|F|}|F^\top|^2 \le \int_{\Sigma_R}(1+|F|)^{-s}.$$

Clearly we have

$$\left|\int_{\partial\Sigma_R}(1+|F|)^{-s}\langle F,\nu\rangle\right| \le \int_{\partial\Sigma_R}(1+|F|)^{1-s}.$$

Combining these two inequalities together we get

$$\int_{\Sigma_R}(1+|F|)^{-s}|\overline{H}|^2 \le (s-n)\int_{\Sigma_R}(1+|F|)^{-s} + \int_{\partial\Sigma_R}(1+|F|)^{1-s}.$$

Choosing $s=n$ yields $(*)$:

$$\int_{\Sigma_R}(1+|F|)^{-n}|\overline{H}|^2 \le \int_{\partial\Sigma_R}(1+|F|)^{1-n}.$$

By our assumption we have that $\exists C>0$ such that for $F(x)=(x,f(x))$ on $\Sigma=\mathbb{R}^n$, we have

$$\det(I+(df)^\top df) \le C$$

on Σ. Since

$$g_{ij} = \delta_{ij} + D_i f^\alpha \cdot D_j f^\alpha,$$

we know that

$$I \le (g_{ij}) \le CI.$$

Hence

$$(1+|x|) \le (1+|F(x)|) \le C(1+|x|).$$

Therefore we get from $(*)$ the key estimate (K):

$$\int_{B_R(0)}(1+|x|)^{-n}|\overline{H}|^2 dx \le C\int_{\partial B_R(0)}(1+|x|)^{1-n} \le C.$$

We now go to the proof of our Theorem.

Proof. Note that the mean curvature flow for the graph of f can be read as

$$\frac{\partial f^\alpha}{\partial t} = g^{ij}D^2_{ij}f^\alpha, \alpha = 1,\cdots,k.$$

The important fact about this equation is that it is invariant under the transformation

$$f(x) \to \frac{1}{\lambda}f(\lambda x), \forall\lambda > 0.$$

Compute

$$
\begin{aligned}
\frac{d}{d\lambda} f_\lambda(x) &= -\lambda^{-2} f(\lambda x) + \lambda^{-1} Df(\lambda x) \cdot x \\
&= \lambda^{-2}[Df(\lambda x) \cdot \lambda x - f(\lambda x)] \\
&= \lambda^{-2} \langle (Df(\lambda x), -1), (\lambda x, f(\lambda x)) \rangle \\
&= \lambda^{-2} \langle (Df(\lambda x), -1), F(\lambda x) \rangle \\
&= \lambda^{-2} \langle (Df(\lambda x), -1), F(\lambda x)^{\perp} \rangle.
\end{aligned}
$$

Here we have used the fact that

$$
(Df(\lambda x), -1) \perp T_p\Sigma.
$$

So

$$
\frac{d}{d\lambda} f_\lambda(x) = \lambda^{-2} \langle (-Df(\lambda x), 1), \overline{H} \rangle.
$$

Hence

$$
\left| \frac{d}{d\lambda} f_\lambda(x) \right| \leq C\lambda^{-2} |\overline{H}|.
$$

So, for $x \in S^{n-1}$, we have

$$
\begin{aligned}
|f_\lambda(x) - f_\mu(x)| &\leq C \int_\lambda^\mu \frac{\overline{H}(\lambda x)}{\sigma^2} d\sigma \\
&\leq C (\int_\lambda^\mu \frac{1}{\sigma^3} d\sigma)(\int_\lambda^\mu \frac{|\overline{H}^2|(\sigma x)}{\sigma} d\sigma) \\
&\leq C|\mu^{-2} - \lambda^{-2}| \int_\lambda^\mu \frac{|\overline{H}(\sigma x)|^2}{\sigma} d\sigma.
\end{aligned}
$$

Notice that, for $\mu \geq \lambda > 1$,

$$
\int_{S^{n-1}} dx \int_\lambda^\mu \frac{|\overline{H}(\sigma x)|^2}{\sigma} d\sigma \leq \int_0^\infty \int_{S^{n-1}} \frac{|\overline{H}(\sigma x)|^2}{(1+\sigma)^n} \sigma^{n-1} dx d\sigma \leq C.
$$

The last inequality follows from the inequality (K). Therefore, we have the estimate $(**)$:

$$
\int_{S^{n-1}} |f_\lambda(x) - f_\mu(x)|^2 dx \leq C|\mu^{-2} - \lambda^{-2}|.
$$

This implies that (f_λ) is a Cauchy sequence in $L^2(S^{n-1})$. Let f_∞ be its unique limit. Since $\sup_{\mathbb{R}^n} |Df_\lambda| = \sup_{\mathbb{R}^n} |Df| \leq C_0$, the Arzela-Ascoli theorem tells us that (f_λ) is compact in $C^\alpha(S^{n-1}), \forall \alpha \in (0,1)$. Therefore

$$
f_\infty(x) = \lim f_\lambda(x) \quad \text{uniformly on } S^{n-1},
$$

and

$$
f_\infty(rx) = r f_\infty(x), \quad \forall 0 \neq r \in \mathbb{R}.
$$

This finishes the proof of Theorem 1.1 □

In the following, we pose a question about the stability of self-similar solutions of (MCF). Let $f_0 : \mathbb{R}^n \to \mathbb{R}^k$ be a smooth function with uniformly bounded (Lipschitz) gradient. Assume

$$
\lim_{\lambda \to \infty} f_{0\lambda} = f_0^\infty, \quad \text{uniformly on } S^{n-1}.
$$

Assume $f : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^k$ such that $F(x,t) = (x, f(x,t))$ is a solution of (MCF) with the initial data $F(x,0) = (x, f_0(x))$. We ask if there is a smooth mapping $\hat{f} : \mathbb{R}^n \to \mathbb{R}^k$ such that $\hat{f}(\cdot, s) \to \hat{f}(\cdot)$ uniformly on compact subsets of $\mathbb{R}^n$ as $s \to \infty$. Here $\hat{f}$ is defined by

$$\hat{f}(x,s) = t^{-\frac{1}{2}} f(\sqrt{t}x, t), s = \frac{1}{2}\log t, 0 \leq s < \infty \text{ with } \quad t \geq 1.$$

A related stability result is done by one of us in [4].

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